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LOWER BOUNDS TO FUNDAMENTAL FREQUENCIES AND BUCKLING LOADS OF COLUMNS AND PLATES†

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Abstract—A general derivation of expressions for lower bounds to fundamental frequencies and buckling loads is given for the class of structures governed by linear elastic theory in the prebuckling state. These expressions involve two Rayleigh quotients both of which are upper bounds for the fundamental frequency under a prescribed load. The displacement trial functions must satisfy force and kinematic continuity but no other conditions are required. Thus, if appropriate high order base functions are used, the finite element procedure can be used to systematically narrow the difference between the upper and lower bounds.

The theory is illustrated with several column and plate problems. The finite element method is applied to uniform and nonuniform columns with a representative set of boundary conditions. Elementary trial functions are used to show that reasonable bounds can also be obtained for plates subjected to known states of stress. Since the lower bound is obtained with a variation of the classical technique of Rayleigh, these results indicate that the method may be suitable for conservatively estimating buckling loads and fundamental frequencies of engineering structures.

1. INTRODUCTION

The theory of elastic stability was initially formulated by Euler and the continued interest in problems of this nature is indicative of the importance of the buckling phenomenon in the design of modern structures. If structural elements cannot be modeled with uniform bending stiffness, regular geometry, or certain boundary conditions, approximate solutions are invariably necessary. Several procedures are available but in the vast majority of cases, the buckling load is computed as a Rayleigh quotient, which is actually an upper bound. This approach violates the basic precept of engineering philosophy that the design of a structure based on an approximate theory should be conservative. Unless the maximum possible error of the upper bound is known, or what amounts to the same thing, unless a lower bound is computed, the use of upper bounds with a factor of safety (unknown) is an untenable situation from the viewpoint of sound engineering practice.

The value of lower bounds to buckling loads for complex structures has been amply demonstrated by the large number of papers on this and the related subject of lower bounds to natural frequencies. The following brief survey is intended to illustrate the approaches that have been attempted with no claim to thoroughness.

In 1935, Weinstein initiated a procedure with slight variations appearing in more recent works. Weinstein's approach, as summarized by Gould[1] or by Weinstein and Stenger[2], consists of exact solutions to a "base" problem with relaxed boundary conditions followed by a sequence of solutions of "intermediate problems" in which the original boundary conditions are progressively enforced with a Galerkin scheme. A variation to this approach in which the physical domain is larger than the original one was introduced by Chi and Mulzet[3]. Constraint conditions are then applied to the complete "extra" domain rather than collocating at points along the physical boundary with the result of better convergence to the exact solution.

Bazley and Fox[4, 5] have expressed what appears to be a similar approach with a more

†This work is based on the dissertation submitted by the first author as partial fulfillment of the requirements for a Ph.D. at the University of New Mexico.

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abstract formulation involving Hilbert space. The positive definite operators inherent in the eigenvalue problem are decomposed in a systematic manner to provide simpler base problems. This work was followed with several papers [6–10] that placed particular emphasis on the natural frequencies of some elastic systems with numerous examples to illustrate the theory.

Fox *et al.*[11] stimulated a sequence of papers with their basic theorem which involved normalized solutions to the governing differential equation. These solutions did not satisfy the boundary conditions so collocation was used to meet the boundary conditions in an approximate sense. Thus, this procedure is also related to that of Weinstein's.

Moler and Payne[12] generalized the initial theorem to the case of all symmetric operators and then developed an additional theorem that provided a bound on eigenfunctions. McLaurin[13] generalized these results somewhat and Nickel[14] considered an approach which allowed approximate solutions to the governing equation as well.

The classical work of Kohn and Koto is described by Crandall[15] but the resulting expression for a lower bound to the lowest eigenvalue requires a reasonable estimate to the second eigenvalue. For buckling, the lowest eigenvalue is the one of interest and the necessity of estimating the second eigenvalue can lead to some difficulty for complex structures.

A new quotient proposed by Nemat-Nasser with a corresponding development for upper and lower bounds [16] on frequencies in layered composites appears to give excellent results and a corresponding theory for buckling loads is expected. For a discrete system, Thompson[17] has developed bounds to buckling loads by considering path derivatives.

Pnueli[18, 19] introduced a particular shift for one of the linear differential operators involved in the buckling equation. This shift makes the problem tractable and provides a lower bound to the buckling load for the original problem. The same operator was decomposed by Masur[20] to provide a norm different from the one that is commonly used in buckling problems and to also provide a system of equations to which solutions could be obtained. Both of these procedures require some insight into the physical nature of the problem and certain criteria must be satisfied.

Generally speaking, the methods utilized in the cited references require the construction of a set of equations that are associated with the actual governing equations, a comparison theorem for the eigenvalues of the two respective systems, and the construction of a suitable finite-dimensional subspace of the underlying Hilbert space. In some cases, exact solutions of the associated system are required. Implicit in the vast majority of cases is the assumption that the prebuckling stress field is known exactly.

From an engineering viewpoint, the construction of an appropriate associated set of equations for a particular class of problems involves considerable ingenuity and the best approach is not obvious for complex structures. Furthermore, exact solutions for those cases where they are required may not be available unless the associated problem is chosen to be quite elementary. This can lead to very poor bounds or a large number of iterations to arrive at a satisfactory answer.

These difficulties are not inherent with the classical Rayleigh quotient [21] which provides upper bounds to buckling loads. As associated system of equations are not required nor are exact solutions to the governing equations. Instead, functions that just satisfy the geometric boundary conditions are required, and coefficients of these functions can be chosen to minimize the quotient. Provided that suitable sequences of functions are used, monotonic convergence to the exact buckling load can be shown. Standard numerical procedures such as the finite element method are directly applicable.

Similar advantages for computing a lower bound are suggested in a theorem given by Isaacson and Keller[22].† This lower bound is given in terms of two upper bounds one of which is the usual Rayleigh quotient. The other upper bound is also given as a quotient so there is the possibility that the theorem can be adapted to continuous systems together with a simplified procedure for obtaining lower bounds.

The method was first applied to a continuous system by Schreyer and Shih[23]. To circumvent certain continuity requirements they formulated the column buckling problem in terms of moment functions but in turn, these functions had to be orthogonal to the self-

†It is not clear who discovered the theorem. See for example, Fox *et al.*[11].

equilibrated functions that are associated with statically indeterminate columns. Popelar[24] presented an elegant generalization in which the two quotients represent the classical Rayleigh and Timoshenko loads and are associated with the principles of minimum potential energy and complementary potential energy, respectively. Although examples involving the fundamental frequency and the buckling of plates are given, the requirement of orthogonality to self-equilibrated stress fields is also inherent in this formulation. Such stress fields are finite in number and specific for columns, but for two-dimensional problems such as plates, an infinite number of these fields can exist, and consequently the requirement of orthogonality between these functions and the set of assumed trial functions is a severe handicap for routine computations of lower bounds. Ku[25, 26] has also used the same basic theorem in conjunction with an innovative function iteration approach. However, since the iterations involve integrations of admissible functions and since a problem dependent matrix must be inverted, it is debatable that this method will prove to be useful for realistic engineering problems.

This paper presents an alternate formulation in which the natural frequency of vibration is expressed in terms of a load parameter. This is a classical approach[27], but the adaptation of the procedure to provide lower bounds to buckling loads is new. The advantage of the approach is that the orthogonality conditions mentioned previously are no longer required. However, there is an accompanying condition that the trial functions satisfy high order continuity requirements. This may be a disadvantage for the finite element method although a recent work [28] indicates that such conditions can be met. In addition to the development of the basic theorem, a preliminary investigation is made of the feasibility of the method with applications to column and plate problems some of which do not have closed form solutions.

2. LOWER BOUNDS BASED ON THE BASIC THEOREM

2.1 *Original theorem*

Since the theorem given in [22] is the basis for this work, it is presented separately in this section. Certain minor modifications to the original presentation have been incorporated so that the correspondence with the development in the next section can be easily shown.

Assume **A** is a self-adjoint positive-definite operator with a compact inverse and solutions to the eigenvalue problem

$$\mathbf{A}\mathbf{y} - \lambda \mathbf{y} = \mathbf{0} \tag{2.1}$$

are denoted by the eigenvectors, $\mathbf{x}_1, \mathbf{x}_2, \ldots$, which are associated with the eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$. Suppose that the eigenvectors have been normalized to satisfy the inner product relation

$$(\mathbf{x}_i; \mathbf{x}_j) = \delta_{ij}. \tag{2.2}$$

Consider a trial vector $\mathbf{x}$. The residual vector, $\boldsymbol{\eta}$, for any value of λ is defined to be

$$\boldsymbol{\eta} = \mathbf{A}\mathbf{x} - \lambda \mathbf{x} \tag{2.3}$$

and thus a measure of the error in the differential eqn (2.1) is given by the positive number F where

$$F^2(\mathbf{x}; \mathbf{x}) = (\boldsymbol{\eta}; \boldsymbol{\eta}). \tag{2.4}$$

The usual Rayleigh quotient, λ_R, and another quotient, $\hat{\lambda}_R$, are defined by choosing λ such that $(\boldsymbol{\eta}; \mathbf{x})$ and $(\boldsymbol{\eta}; \mathbf{A}\mathbf{x})$ are zero, respectively. The result is

$$\lambda_R = \frac{(\mathbf{A}\mathbf{x}; \mathbf{x})}{(\mathbf{x}; \mathbf{x})} \tag{2.5a}$$

$$\hat{\lambda}_R = \frac{(\mathbf{A}\mathbf{x}; \mathbf{A}\mathbf{x})}{(\mathbf{A}\mathbf{x}; \mathbf{x})}. \tag{2.5b}$$

For the sake of convenience, a third quotient, λ_S, is introduced by the following relation:

$$\lambda_S^2 = \lambda_R \hat{\lambda}_R = \frac{(\mathbf{Ax}; \mathbf{Ax})}{(\mathbf{x}; \mathbf{x})}. \tag{2.6}$$

This function is merely a second Rayleigh quotient associated with the eigenvalue problem

$$\mathbf{A}^2\mathbf{y} - \lambda^2\mathbf{y} = \mathbf{0} \tag{2.7}$$

with eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots$, and eigenvalues $\lambda_1^2 \leq \lambda_2^2 \leq \ldots$.

The eigenvectors $\mathbf{x}_i$ represent a complete basis for the infinite dimensional space so that any vector $\mathbf{x}$ can be represented by the sum

$$\mathbf{x} = \sum_i a_i \mathbf{x}_i \tag{2.8}$$

in which a_i denote constants.

The use of the definition of λ_i and the substitution of eqns (2.2) and (2.8) into eqn (2.4) yields

$$\begin{aligned} F^2 \sum_j a_j^2 &= \sum_i a_i^2(\lambda_i^2 - 2\lambda\lambda_i + \lambda^2) \\ &= \sum_i a_i^2(\lambda_i - \lambda)^2 \\ &\geq \min_i (\lambda_i - \lambda)^2 \sum_j a_j^2 \end{aligned} \tag{2.9}$$

and hence

$$F^2 \geq \min_i (\lambda_i - \lambda)^2. \tag{2.10}$$

Suppose λ is sufficiently close to λ_1 and define λ_L to be

$$\lambda_L = \lambda - F. \tag{2.11}$$

Then it follows from eqn (2.10) that

$$\lambda_L \leq \lambda_1 \tag{2.12}$$

i.e. λ_L is a lower bound to the lowest eigenvalue. The Cauchy–Schwarz inequality can be used to show the upper bound relations

$$\lambda_1 \leq \lambda_R \leq \lambda_S \leq \hat{\lambda}_R. \tag{2.13}$$

The substitution of eqns (2.5a) and (2.6) into eqn (2.4) also yields

$$F^2 = \lambda_S^2 - 2\lambda\lambda_R + \lambda^2. \tag{2.14}$$

The requirement that λ be sufficiently close to λ_1 can now be given explicity as the weaker of the two relations

$$\lambda \leq \frac{1}{2}(\lambda_1 + \lambda_2) \tag{2.15a}$$

$$\lambda \leq \frac{1}{2}(\lambda_S^2 - \lambda_1^2)/(\lambda_R - \lambda_1). \tag{2.15b}$$

The first is a statement that λ must be closer to λ_1 than any other eigenvalue [22] while the second results from a consideration of eqns (2.10) and (2.14). Ku[25] was apparently the first to recognize that for many cases eqn (2.15b) is not nearly as restrictive as eqn (2.15a).

Since λ_1 and λ_2 are not known, suppose λ is chosen to minimize F^2. Then

$$\lambda = \lambda_R \tag{2.16a}$$

$$(F^2)_{\min} = \lambda_S^2 - \lambda_R^2 \tag{2.16b}$$

and from eqn (2.11)

$$\lambda_L = \lambda_R - [\lambda_S^2 - \lambda_R^2]^{1/2}. \tag{2.17}$$

Equation (2.17) is an explicit relation for a lower bound in terms of two upper bounds that are normally easy to compute. The requirements that $\lambda = \lambda_R$ and that λ satisfy eqns (2.15a) or (2.15b) are compatible conditions since λ_R is usually a very good upper bound to λ_1 for a reasonable choice of **x**.

If **x** is an approximation to $\mathbf{x}_1$ that is accurate to first order in small parameter ϵ, then λ_R and λ_S represent approximations to λ_1 that are accurate to second order in ϵ. It follows from eqn (2.17) that λ_L is an approximation to λ_1 that is accurate to first order in ϵ. Thus, for reasonable trial functions, the upper bounds will be better approximations to λ_1 than the lower bound.

2.2 *Extension to a more general problem*

Although a study of the eigenvalue problem of eqn (2.1) is of considerable interest, this equation is not directly applicable to problems of elastic stability. Instead, the eigenvalue problem

$$\mathbf{Ay} - \lambda \mathbf{By} = 0 \tag{2.18}$$

in which **A** and **B** are assumed to be positive definite operators which are not adjoint, is more appropriate to the determination of lower bounds to elastic buckling loads. For two different eigenvectors, $\mathbf{y}_\alpha$ and $\mathbf{y}_\beta$, the orthogonality condition, $(By_\alpha; By_\beta) = \delta_{\alpha\beta}$, is not satisfied and consequently a direct development of a lower bound theorem from eqn (2.18) is not possible.

An alternate formulation that circumvents this difficulty is to define the operator

$$\hat{\mathbf{A}} = \mathbf{A} - \lambda \mathbf{B} \tag{2.19}$$

which is positive definite for small enough λ and which is assumed to be self-adjoint. Now consider the associated problem

$$\hat{\mathbf{A}}\mathbf{y} - \Lambda \mathbf{y} = 0 \tag{2.20}$$

for fixed λ.† The theory of the previous section can be applied directly to obtain the Rayleigh quotients and the lower bounds

$$\Lambda_R(\lambda) = \frac{(\hat{\mathbf{A}}\mathbf{x}; \mathbf{x})}{(\mathbf{x}; \mathbf{x})} \tag{2.21a}$$

$$\Lambda_S^2(\lambda) = \frac{(\hat{\mathbf{A}}\mathbf{x}; \hat{\mathbf{A}}\mathbf{x})}{(\mathbf{x}; \mathbf{x})} \tag{2.21b}$$

$$\Lambda_L(\lambda) = \Lambda_R - (\Lambda_S^2 - \Lambda_R^2)^{1/2} \tag{2.21c}$$

in which the dependence on λ has been purposely emphasized, and for any value of λ, the

†For a continuous medium Λ is proportional to the square of the natural frequency.

following inequality holds:

$$\Lambda_L \leq \Lambda_1 \leq \Lambda_R \leq \Lambda_S. \tag{2.22}$$

Implicit in the formulation is the assumption that a trial function must satisfy the required continuity and boundary conditions† associated with the quotient Λ_S^2. Thus, for certain problems in which the boundary conditions involve λ, the trial functions may be a function of λ.

Consider the class of problems for which the boundary conditions and hence the trial vectors do not depend on λ. Then Λ_R decreases linearly with λ and Λ_S^2, which is positive, semi-definite in λ, initially descreases, and then increases quadratically with λ. As shown in Fig. 1, define λ_L and λ_R as the values of λ for which $\Lambda_L = 0$ and $\Lambda_R = 0$, respectively. It follows from eqns (2.21) and (2.22) that

$$\lambda_L \leq \lambda_1 \leq \lambda_R \tag{2.23}$$

which is the desired inequality.

For cases in which trial vectors involve λ, the behavior of the Λ's on λ is more complicated and it is not obvious that the same general properties should follow. However, the inequalities of eqns (2.22) and (2.23) are satisfied and the bounds provide nontrivial results for all such sample problems considered in this investigation.

2.3 *A numerical procedure*

For almost any buckling problem of engineering interest, and, in particular, for problems used in this study to illustrate the effectiveness of the procedure, the exact formulation must be replaced with a finite dimensional eigenvalue problem if an approximate solution is to be obtained. This leads to an eigenvalue problem of the type.

$$\mathbf{Cz} - \Lambda\mathbf{Dz} = \mathbf{0} \tag{2.24}$$

in which Λ denotes the eigenvalue. The eigenvector is given by $\mathbf{z}$, $\mathbf{D}$ is a positive definite matrix while $\mathbf{C}$ may be positive semi-definite. An efficient iterative scheme for obtaining a particular eigenvalue is to use the power method, that is to solve the set of equations

$$[\mathbf{C} - (\Lambda^{(n-1)} - 1)\mathbf{D}]\mathbf{z}^{(n)} = \mathbf{Dz}^{(n-1)}; \quad n = 1, 2, \ldots \tag{2.25a}$$

$$\Lambda^{(n)} = \frac{(\mathbf{z}^{(n)}; \mathbf{Cz}^{(n)})}{(\mathbf{z}^{(n)}; \mathbf{Dz}^{(n)})} \tag{2.25b}$$

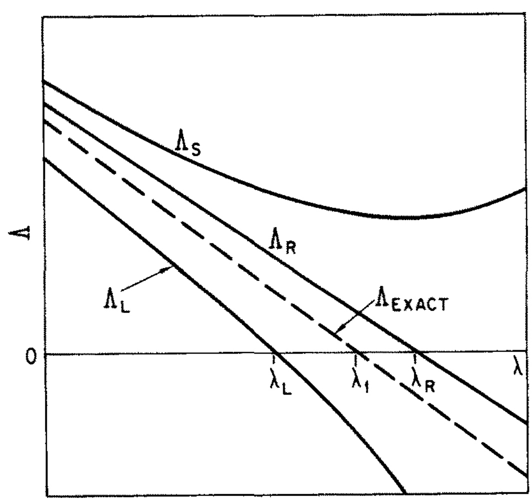

Fig. 1. General behavior of Rayleigh quotients and Λ_L as a function of λ.

†Of course, only the kinematic conditions must be satisfied if Λ_R is to be computed independently of this lower bound procedure.

in which a semi-colon denotes an inner product and with $\Lambda^{(0)}$ and $\mathbf{z}^{(0)}$ chosen reasonably to minimize the number of iterations. Gaussian elimination was used as a convenient method for solving the matrix equation although other schemes may be more efficient for large scale routine applications. The shift in the eigenvalue spectrum is introduced since a zero eigenvalue is of interest. The iteration is terminated when both of the conditions

$$\left| \|\mathbf{z}\|^{(n)} - \|\mathbf{z}\|^{(n-1)} \right| \le \epsilon_1 \tag{2.26a}$$

$$|\Lambda^{(n)} - \Lambda^{(n-1)}| \le \epsilon_2 \tag{2.26b}$$

are satisfied where ϵ_1 and ϵ_2 are suitably chosen positive constants and $\|\mathbf{z}\|$ denotes a norm of the vector $\mathbf{z}$.

For this application, the matrix eigenvalue problem can be formulated by minimizing either Λ_R or Λ_S^2. For either choice, a solution must be obtained for a sequence of values of λ so that the zero of Λ_L and Λ_R can be obtained. To minimize the number of eigenvalue evaluations, an algorithm such as the secant method can be used.

3. APPLICATIONS TO COLUMNS

The basic equations of the previous section can be readily transformed into a notation that is more commonly used in continuum mechanics. Since such a development can be rather routinely performed, the primary emphasis of this and the next section is placed on illustrating the use of the basic theorem. Common structural elements are used for this purpose.

3.1 *Theory*

For a column of length L, the Kirchhoff and Euler assumptions yield the beam equation for free vibration

$$V' - \Lambda w = 0 \tag{3.1}$$

in which w denotes the lateral displacement. If ω is the natural frequency and μ the mass density per unit length, then

$$\Lambda = \mu L^4 \omega^2. \tag{3.2}$$

The prime denotes a derivative with respect to the dimensionless variable, x, with domain [0, 1]. The transverse shear, V, and bending moment, M, are related to w as follows:

$$V = M' + \lambda P w' \tag{3.3a}$$

$$M = EIw''. \tag{3.3b}$$

The bending stiffness, EI, and the "unit" axial force multiplied by L^2, which is denoted by P, are functions of x. The magnitude of the load is represented by the parameter λ. The boundary conditions associated with the governing differential equation are

$$w = 0 \text{ or } V = 0 \text{ and } w' = 0 \text{ or } M = 0 \text{ at } x = 0, 1. \tag{3.4}$$

An application of the theory given in the previous section yields the following forms for the Rayleigh quotients:

$$\Lambda_R = \int_0^1 [(EIw''w'') - \lambda P(w')^2]\,dx \Big/ \int_0^1 w^2 dx \tag{3.5a}$$

$$\Lambda_S^2 = \int_0^1 (V')^2\,dx \Big/ \int_0^1 w^2\,dx. \tag{3.5b}$$

It follows that any trial function, w, as well as w', M and V must be continuous, and both the kinematic and the force boundary conditions must be satisfied.

In order to sequentially improve the accuracy for the upper and lower bounds of the natural frequency for a given axial load, a procedure for consistently improving the trial function is required. There are several methods that can be applied. Perhaps the most classical approach is to assume a free vibration mode composed of prescribed functions that satisfy the boundary conditions with parameters that are chosen by minimizing one of the Rayleigh quadratic forms.

Another method of systematically improving the trial functions for the free vibration mode is to use finite elements with the base functions for each element chosen so that the force and kinematic continuity conditions are automatically satisfied. The base functions and matrix development used for this portion of the study are given in the Appendix.

3.2 *Numerical results*

The analyses of several column problems were performed with the parameters w, w', M and V at the end of each element defined as the components of $\mathbf{z}$ in eqns (2.25a) and (2.25b). As expected, if the bounds are determined by minimizing the first Rayleigh quotient of eqn (3.5a) the values of Λ_R and λ_R decrease monotonically with an increase in the number of elements. Generally speaking, Λ_S, Λ_L and λ_L improve when N is increased but this behavior is not guaranteed. An alternate procedure that was used in this study and which appears to provide closer bounds is to minimize the second Rayleigh quotient and obtain corresponding values for Λ_R, Λ_L, λ_R and λ_L.

Typical numerical results for Rayleigh and lower bounds of buckling loads as functions of N are given in Tables 1–3. The bending stiffness distributions are given directly on the tables for columns of unit length.

These results indicate a considerable improvement in the bounds given previously by Schreyer and Shih[23] for the same problems. Table 1 shows the rate of convergence for simply-supported and fixed-free lateral uniform columns, both of which have an exact buckling load of 9.8696. Tables 2 and 3 provide similar results for nonuniform fixed-free, fixed-simply supported and fixed-fixed columns for which exact buckling loads are not available. The best

Table 1. Numerical results for uniform columns

N	Simply-Supported		Fixed-Free Lateral	
	EI = 1, $0 \le x \le 1$		EI = 1, $0 \le x \le 1$	
	λ_R	λ_L	λ_R	λ_L
1	9.8696	9.6446	9.8696	9.8063
2	9.8696	9.8582	9.8696	9.8499
3	9.8696	9.8669	9.8696	9.8639
4	9.8696	9.8653	9.8696	9.8641

Table 2. Numerical results for nonuniform columns

N	Fixed-Free		Fixed-Simply Supported	
	EI = 2, $0 \le x \le 1/2$ EI = 1, $1/2 < x \le 1$		EI = 2, $0 \le x \le 1/2$ EI = 1, $1/2 < x \le 1$	
	λ_R	λ_L	λ_R	λ_L
2	4.1345	4.1335	25.1831	24.9355
3	4.1345	4.1312	25.1831	24.9361
4	4.1345	4.1275	25.1831	25.1678

Table 3. Numerical results for nonuniform fixed–fixed columns

N	EI = 2, 0 ≤ x ≤ 1/4; EI = 1.7, 1/4 < x ≤ 1/2; EI = 1.3, 1/2 < x ≤ 3/4; EI = 1, 3/4 < x ≤ 1		EI = 2, 0 ≤ x ≤ 1/4; EI = 1.5, 1/4 < x ≤ 3/4; EI = 1, 3/4 < x ≤ 1	
	λ_R	λ_L	λ_R	λ_L
4	56.3010	56.1435	57.9753	57.8192
6	56.3010	56.2010	57.9753	57.8895
8	56.3010	56.2499	57.9753	57.9250

lower bound for each column differs from the upper bound by less then 0.1% of the upper bound.

4. APPLICATIONS TO RECTANGULAR PLATES

4.1 *Theory*

For an isotropic, uniform plate subjected to in-plane forces, von Karman's equation for free vibration is

$$D\nabla^4 w + \lambda N^0_{\alpha\beta} w_{,\alpha\beta} - \Lambda w = 0 \tag{4.1}$$

in which w denotes the lateral displacement, and

$$\Lambda = \mu\omega^2 \tag{4.2a}$$

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{4.2b}$$

where Young's modulus, Poisson's ratio, the plate thickness and the mass density per unit area are given by E, ν, h and μ, respectively. The negative of $N^0_{\alpha\beta}$ represents the prebuckling membrane forces for a "unit load". In this section Greek indices assume the values 1 and 2 and the summation convention is used.

For convenience, let **n** denote a unit vector normal to the edge of the plate and associate this vector with a normal cartesian coordinate n. Similarly, consider a locally cartesian coordinate s with tangent vector s directed in the counter-clockwise direction around the edge of the plate for any point along the edge. Furthermore let,

$$V_\gamma = D[(1-\nu)\delta_{\alpha\gamma}\delta_{\beta\rho} + \nu\delta_{\alpha\beta}\delta_{\gamma\rho}]\ w_{,\alpha\beta\rho} + \lambda N^0_{\alpha\gamma} w_{,\alpha} \tag{4.3a}$$

$$M_n = D[(1-\nu)\delta_{\alpha\gamma}\delta_{\beta\rho} + \nu\delta_{\alpha\beta}\delta_{\gamma\rho}]\ w_{,\alpha\beta} n_\rho n_\gamma \tag{4.3b}$$

$$M_{ns} = D[(1-\nu)\delta_{\alpha\gamma}\delta_{\beta\rho} + \nu\delta_{\alpha\beta}\delta_{\gamma\rho}]\ w_{,\alpha\beta} n_\rho n_\gamma. \tag{4.3c}$$

Then eqn (4.1) becomes

$$V_{\gamma,\gamma} - \Lambda w = 0 \text{ on } R \tag{4.4}$$

with the boundary conditions

$$\left.\begin{aligned} V_\gamma n_\gamma + M_{ns,s} = 0 \text{ or } w = 0 \\ M_n = 0 \text{ or } w_{,n} = 0 \end{aligned}\right\} \text{on } \partial R \tag{4.5}$$

and the requirement that M_{ns} or w be continuous at each corner.

The Rayleigh quotients for this class of problems become

$$\Lambda_R = \frac{\int_R [D\{(1-\nu)\delta_{\alpha\gamma}\delta_{\beta\rho} + \nu\delta_{\alpha\beta}\delta_{\gamma\rho}\}w_{,\alpha\beta}w_{,\gamma\rho} - \lambda N^0_{\alpha\beta}w_{,\alpha}w_{,\beta}]\,\mathrm{d}A_p}{\int_R w^2\,\mathrm{d}A_p} \tag{4.6a}$$

$$\Lambda_S^2 = \frac{\int_R V_{\alpha,\alpha}V_{\beta,\beta}\,\mathrm{d}A_p}{\int_R w^2\,\mathrm{d}A_p} \tag{4.6b}$$

in which $\mathrm{d}A_p$ denotes a plate area element for the region R.

4.2 *Sample results*

For the sake of definiteness, eqns (4.6a), (4.6b) and (2.21c) are used to obtain Rayleigh quotients and lower bounds for five simple problems. For each case $D = 1$, $\nu = 0.3$ and the plate is a unit square defined by $0 \le x \le 1$, $0 \le y \le 1$.

The first three cases represent plates under uniform compression, i.e. $N^0_{\alpha\beta} = \delta_{\alpha\beta}$ while the last two involve slight generalizations to nonuniform stress. Although the exact displacement function is known in the first case, a polynomial function that satisfies the boundary conditions is used to illustrate that nontrivial (greater than zero) lower bounds can be obtained. The results are summarized as follows:

Problem 1. *Plate simply supported on all sides*

$$w = a(x^4 - 2x^3 + x)(y^4 - 2y^3 + y)$$

$$\lambda_R = 19.8 \qquad \lambda_L = 16.5$$

$$\lambda_1 = 2\pi^2 = 19.7 \quad \text{[Ref. 21]}.$$

Problem 2. *Plate clamped on all sides.*

(a)

$$w = a(x^4 - 2x^3 + x^2)(y^4 - 2y^3 + y^2)$$

$$\lambda_R = 54.0 \qquad \lambda_L = 19.1.$$

(b)

$$w = a(1 - \cos 2\pi x)(1 - \cos 2\pi y)$$

$$\lambda_R = 52.6 \qquad \lambda_L = 28.0$$

$$\lambda_1 = 5.30\pi^2 = 52.3 \quad \text{[Refs. 21 and 29]}.$$

Problem 3. *Plate simply supported on two opposite sides and clamped on the other two sides*

$$w = a(x^4 - 2x^3 + x)(y^4 - 2y^3 + y^2)$$

$$\lambda_R = 38.5 \qquad \lambda_L = 20.8$$

$$\lambda_1 = 3.83\pi^2 = 37.8 \quad \text{[Ref. 29]}.$$

Problem 4. *Plate simply supported on all sides with nonuniform stress field.* The displacement and prebuckling stress fields are given as follows:

$$w = a(x^4 - 2x^3 + x)(y^4 - 2y^3 + y)$$

$$N_{11} = \begin{cases} 1 + \epsilon - 4\epsilon x & \text{for } 0 \le x < \frac{1}{2} \\ 1 - 3\epsilon + 4\epsilon x & \text{for } \frac{1}{2} \le x \le 1 \end{cases}$$

$$N_{22} = \begin{cases} 1 + \epsilon - 4\epsilon y & \text{for } 0 \le y < \frac{1}{2} \\ 1 - 3\epsilon + 4\epsilon y & \text{for } \frac{1}{2} \le y \le 1 \end{cases}$$

$$N_{12} = N_{21} = 0$$

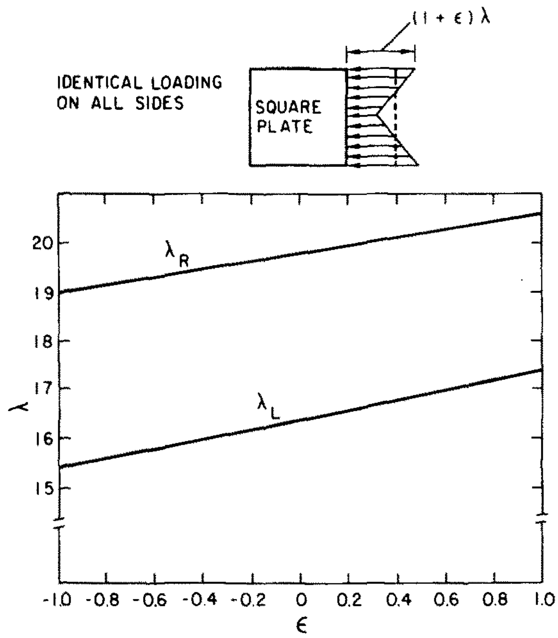

Fig. 2. Upper and lower bounds for buckling loads of a simply supported plate with nonuniform stress field.

where the constant, ϵ, is a stress distribution parameter. Lower and upper bounds as a function of ϵ are shown in Fig. 2. The fact that both the upper and lower bounds decrease with negative ϵ is highly significant in that this is an indication that if the prebuckling stresses are not known exactly, computed buckling loads based on a uniform but statically equivalent prebuckling stress field may not be conservative.

Problem 5. *Two opposite edges simply supported and two edges free with unequal but uniform prebuckling stress components.* Consider a plate for which the edges $x = 0$ and $x = 1$ are simply supported and the other two edges ($y = 0$ and $y = 1$) are free. The stress field is assumed to be $N_{22} = 1$, $N_{12} = 0$ and $N_{11} = \kappa$ where the parameter κ is a constant. Let

$$w(x, y) = af(y) \sin \pi x$$

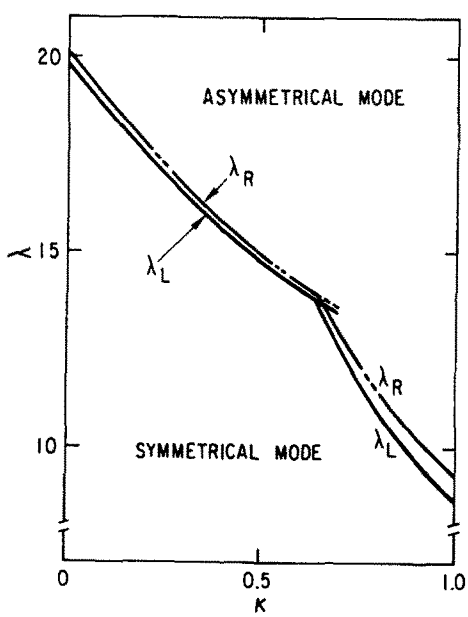

Fig. 3. Upper and lower bounds for buckling loads of a plate with two opposite edges simply supported and two free edges.

in which

$$f(y) = 1 + \sum_{n=1}^{5} a_n y^n$$

is the lowest order polynomial that is admissible. For a given value of λ, the coefficients a_n are chosen to satisfy the conditions of zero moment and shear on the free edges in addition to the requirement of $f(0) = f(1)$ for a symmetrical buckling mode or $f(0) = -f(1)$ for an antisymmetrical mode. The results are shown in Fig. 3 which shows plots of λ_R and λ_L as a function of κ. The closeness of the bounds is merely a reflection of the fact that the assumed displacement field accurately reflects the characteristics of the exact solution.

Although these plate problems are somewhat elementary, they do illustrate that this lower bound procedure can provide reasonable, and in some cases, very good results. The underlying reason for the significantly poorer lower bounds for problems involving fixed boundaries is not known and further investigations in this area would be worthwhile. The use of higher order elements such as those suggested by Gopalacharyulu[28] would provide the necessary continuity requirements and make the procedure amendable to a systematic finite element analysis for plates with a variable stress field and arbitrary boundaries.

5. SUMMARY AND CONCLUSIONS

With the use of a natural frequency parameter, the Isaacson–Keller lower bound theorem has been extended to include buckling problems of elastic structures. Trial functions can be systematically improved with a procedure involving finite elements. Bounds on the buckling load are obtained with a formulation of minimizing either the first or second Rayleigh quotient. However, the minimization of one Rayleigh quotient does not guarantee monotonic improvement for the other Rayleigh quotient and the lower bound, although these latter two parameters generally do get better. Round-off error also causes problems when a numerical procedure with a large number of elements is used.

The theorem is applicable provided the Rayleigh upper bound is sufficiently close to the fundamental frequency. Since simplified problems can be used as a guide for the choice of an approximation to the buckling mode, this requirement offers no practical barrier to the use of the theorem. Of much more serious consequence is the requirement that the force boundary and continuity conditions be satisfied. Also, it has been implicitly assumed that the exact functions for the prebuckling stresses are known. If the theorem is to become a routine analytical procedure, it is essential that these conditions be relaxed. Further research on these aspects would seem to be particularly appropriate. Any advancement in this regard might also provide some insight into the reason for the poor lower bounds for clamped-edged plates.

In spite of these restrictions, there are several advantages associated with the procedure in its present form. The use of the theorem is simple, solutions of intermediate problems are not required, and the lower bound converges to the buckling load although at a slower rate than the Rayleigh upper bounds. The feasibility of the theorem for engineering structures has been demonstrated with the analyses of column and plate problems. In most cases, the lower bounds are close enough to the upper bounds to serve as useful design values. Furthermore, it is believed that this work provides the basis for the development of equations so that routine calculations can be made to obtain lower bounds to buckling loads and fundamental frequencies for more complex engineering structures.

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APPENDIX

The development of finite element equations for the analysis of columns is based on the following displacement field for each element:

$$\begin{aligned} w = w_L f_1(y) + h w_L'\left[f_2(y) - \frac{\lambda P h^2}{EI}\frac{f_4(y)}{768}\right] + \frac{h^2}{EI}\frac{M_L}{21} f_3(y) \\ + \frac{h^3}{EI}\frac{V_L}{768} f_4(y) + w_R f_5(y) + h w_R'\left[f_6(y) - \frac{\lambda P h^2}{EI}\frac{f_8(y)}{768}\right] \\ + \frac{h^2}{EI}\frac{M_R}{21} f_7(y) + \frac{h^3}{EI}\frac{V_R}{768} f_8(y). \end{aligned} \tag{A1}$$

The nondimensional length of the element is h, and y has the range $[0, h]$; w_L, w_L', M_L, V_L, w_R, w_R', M_R and V_R' denote the displacement, slope, bending moment and shear force at the left and right ends of an element, respectively. Consequently the kinematic and force continuity and boundary conditions are automatically satisfied. The base functions are:

$$f_1(y) = 1 - 35\left(\frac{y}{h}\right)^4 + 84\left(\frac{y}{h}\right)^5 - 70\left(\frac{y}{h}\right)^6 + 20\left(\frac{y}{h}\right)^7$$

$$f_2(y) = \frac{y}{h} - 20\left(\frac{y}{h}\right)^4 - 45\left(\frac{y}{h}\right)^5 - 36\left(\frac{y}{h}\right)^6 + 10\left(\frac{y}{h}\right)^7$$

$$f_3(y) = 21\left[\frac{1}{2}\left(\frac{y}{h}\right)^2 - 5\left(\frac{y}{h}\right)^4 + 10\left(\frac{y}{h}\right)^5 - \frac{15}{2}\left(\frac{y}{h}\right)^6 + 2\left(\frac{y}{h}\right)^7\right]$$

$$f_4(y) = 768\left[\frac{1}{6}\left(\frac{y}{h}\right)^3 - \frac{2}{3}\left(\frac{y}{h}\right)^4 + \left(\frac{y}{h}\right)^5 - \frac{2}{3}\left(\frac{y}{h}\right)^6 + \frac{1}{6}\left(\frac{y}{h}\right)^7\right]$$

$$f_5(y) = 35\left(\frac{y}{h}\right)^4 - 84\left(\frac{y}{h}\right)^5 + 70\left(\frac{y}{h}\right)^6 - 20\left(\frac{y}{h}\right)^7$$

$$f_6(y) = -15\left(\frac{y}{h}\right)^4 + 39\left(\frac{y}{h}\right)^5 - 34\left(\frac{y}{h}\right)^6 + 10\left(\frac{y}{h}\right)^7$$

$$f_7(y) = 21\left[\frac{5}{2}\left(\frac{y}{h}\right)^4 - 7\left(\frac{y}{h}\right)^5 + \frac{13}{2}\left(\frac{y}{h}\right)^6 - 2\left(\frac{y}{h}\right)^7\right]$$

$$f_8(y) = 768\left[-\frac{1}{6}\left(\frac{y}{h}\right)^4 + \frac{1}{2}\left(\frac{y}{h}\right)^5 - \frac{1}{2}\left(\frac{y}{h}\right)^6 + \frac{1}{6}\left(\frac{y}{h}\right)^7\right]$$

in which scaling constants have been introduced to yield inner products of the same order of magnitude for all pairs.

For each element, let the components of matrices $\mathbf{K}^1, \ldots, \mathbf{K}^5$ be defined as follows:

$$\begin{aligned}
K^1_{ij} &= \frac{1}{h^8}\int_0^1 f''''_i f''''_j \,\mathrm{d}y \\
K^2_{ij} &= \frac{1}{2h^6}\int_0^1 (f''''_i f''_j + f''_i f''''_j)\,\mathrm{d}y \\
K^3_{ij} &= \frac{1}{h^4}\int_0^1 f''_i f''_j \,\mathrm{d}y \\
K^4_{ij} &= \frac{1}{h_2}\int_0^1 f'_i f'_j \,\mathrm{d}y \\
K^5_{ij} &= \int_0^1 f_i f_j \,\mathrm{d}y \qquad i, j = 1, \ldots 8.
\end{aligned} \tag{A3}$$

Then the required integrals for each element become

$$\begin{aligned}
&\int_e [(EIw'')'' + \lambda P w'']^2 \,\mathrm{d}x = \mathbf{z}^T[(EI)^2\mathbf{K}^1 + 2EI\lambda P\mathbf{K}^2 + (\lambda P)^2\mathbf{K}^3]\mathbf{z} \\
&\int_e [(EIw''w'') - \lambda P w' w']\,\mathrm{d}x = \mathbf{z}^T[EI\mathbf{K}^3 + \lambda P\mathbf{K}^4]\mathbf{z} \\
&\int_e w^2\,\mathrm{d}x = \mathbf{z}^T\mathbf{K}^5\mathbf{z}
\end{aligned} \tag{A4}$$

in which

$$\mathbf{z} = \left(w_L, w'_L, \frac{M_L}{21}, \frac{V_L}{768}, w_R, w'_R, \frac{M_R}{21}, \frac{V_R}{768}\right)^T \tag{A5}$$

EI and P are assumed to be constant over each element. The integrals for the complete column are obtained by a simple summation to include all elements.